Ultrashort-Pulse Child-Langmuir Law in the Quantum and Relativistic Regimes

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This Letter presents a consistent quantum and relativistic model of short-pulse Child-Langmuir (CL) law, of which the pulse length τ is less than the electron transit time in a gap of spacing D and voltage V. The classical value of the short-pulse CL law is enhanced by a large factor due to quantum effects when the pulse length and the size of the beam are, respectively, in femtosecond duration and nanometer scale. At high voltage larger than the electron rest mass, relativistic effects will suppress the enhancement of short-pulse CL law, which is confirmed by particle-in-cell simulation. When the pulse length is much shorter than the gap transit time, the current density is proportional to V, and to the inverse power of D and τ.

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Laser-driven short-pulse electron beam is a crucial component in various areas, such as photoinjectors for free electron lasers (FEL) [1], laser acceleration of relativistic electrons in free space [2], ultrafast electron microscopy [3], picosecond cathodoluminescence [4], and femtosecond electron diffraction [5]. Using a low power femtosecond laser irradiation on a sharp tungsten tip (down to 2 nm in diameter), ultrashort electron pulse from 70 fs to less than 1 fs duration was produced by using optical field emission at an optical electric field of >1 GV/m [6,7]. The local current intensity of such ultrashort electron pulse is about 15 kA/cm^2 or more [7].

For laser-driven photocathodes used in FEL, the electron currents of 0.1 to 1 nC bunches in 10–50 ps with applied dc fields of 10–50 MV/m are produced through photoemission by using low work-function coated metal surface [8]. In general, photoemission based cathodes are normally operated in source-limited regime. To extract as much current as possible from a photocathode, space charge effects will become important at high current operation [9]. For space charge limited (SCL) emission, the maximum steady-state current density that can be transported across a gap of spacing D and potential difference V_g is described by the familiar one-dimensional (1D) Child-Langmuir (CL) law [10],

$$J_{\mathrm{CL}} = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V_g^{3/2}}{D^2}, \quad (1)$$

where e and m are the charge and mass of the electron, respectively, and ϵ_0 is the free space permittivity. While Eq. (1) is easy to derive, it was only recently that the 1D steady-state classical CL law was extended to multidimensional models [11–14] and quantum regime [15,16].

When the pulse length (τ) of the electron beam is less than the gap transit time, Eq. (1) is no longer valid, and a classical short-pulse CL law has been derived [9], which shows that the classical value of the steady-state (or long-pulse) CL law is enhanced by a factor given by

$$\frac{J_{\mathrm{SCL}}}{J_{\mathrm{CL}}} = 2\frac{1-\sqrt{1-3X_{\mathrm{CL}}^2/4}}{X_{\mathrm{CL}}^3}, \quad (2)$$

where $X_{\mathrm{CL}} = \tau/T_{\mathrm{CL}} \le 1$ is the normalized transit time and $T_{\mathrm{CL}} = 3D/\sqrt{2eV_g/m}$ is the gap transit time of the 1D classical CL law. However, Eq. (2) is not valid at ultrashort time scale (like 1 ps or less), for which the size of the electron pulse is comparable to the electron de Broglie wavelength (near to the cathode). As an example, consider a femtosecond laser irradiation on a sharp emitter tip with a local dc field of $E = 0.2$ GV/m [7], the emitted short-pulse electron beam (without space charge effects) is estimated to have a characteristic length of $\delta x \sim \frac{e}{m}E\delta t^2 \sim 2$ nm with a pulse length of $\delta t = 7$ fs. For a typical photoinjector at a dc field of 10 MV/m, the characteristic length is about $\delta x \sim 18$ nm if a 100 fs electron bunch is assumed. Since the length scale is in nanometer regime, quantum effects of the electron transport near to the cathode are important [15,16]. In additional to the quantum effects, relativistic effects may also become important if the electron bunches are accelerated to MV range inside the gap.

Thus, the 1D classical short-pulse CL law shown in Eq. (2) requires a complete revision to include quantum and relativistic effects. In this context, we speculate that the recent advances in ultrafast electron bunches may be operated in high current regime, where space charge effects cannot be ignored. It is of interests to develop a simple 1D model of ultrashort-pulse CL law to account for the essential quantum and relativistic effects over a wide range of applied voltage, gap spacing, and pulse duration.

For simplicity, we consider a 1D planar diode of gap spacing D, with a grounded cathode, and a dc potential of V_g is applied to the anode. A uniform electron beam of a current density J with a finite pulse length τ is injected normally into the vacuum gap from the cathode. Here, we assume unlimited electron current can be supplied and ignore the process of the electron emission at the cathode (either photoemission or optical field emission). Since the pulse length τ is shorter than the gap transit time T_{CL}, the electron pulse can only extend to a distance of ζ, which is smaller than the gap spacing D, and T_{CL} is a function of V_g and D determined at the SCL condition in the respective quantum and relativistic regime. At the beam-vacuum interface (or beam front) of $x = \zeta$, the electric potential field is defined as ϕ_ζ with a continuous electric field. In this 1D model, we calculate the maximum current density that can be transported (before the formation of virtual cathode) as a function of V_g, D, and τ.

To obtain the short-pulse quantum SCL current density $J = J_{\mathrm{QCL}}$, the beam propagation region of $0 \leq x \leq \zeta$ (during the pulse length τ), which has a potential difference of ϕ_ζ, is considered as an equivalent quantum diode based thin-sheet model, and J_{QCL} is formulated as

$$J_{\mathrm{QCL}} \equiv \mu_\zeta \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{\phi_\zeta^{3/2}}{\zeta^2} = \frac{a_\zeta - c_\zeta}{\tau}\frac{\phi_\zeta}{\zeta}. \tag{3}$$

Here, μ_ζ is the quantum enhancement factor, a_ζ and c_ζ are, respectively, the normalized electric field at the beam front ($x = \zeta$), and at the cathode ($x = 0$), which are calculated by solving the 1D time-independent Schrödinger equation, the Poisson equation, and charge conservation relation in a mean field model [15]. The coupled equations (in normalized form) are

$$q'' + \lambda^2\left(\phi - \frac{V_{xc}}{\phi_g} - \frac{4}{9}\frac{\mu_\zeta^2}{q^4}\right)q = 0, \tag{4}$$

$$\phi'' = \frac{2}{3}q^2, \tag{5}$$

where the prime denotes the derivatives with respect to $\bar{x} = x/\zeta$, λ measures the ratio of ζ to the electron de Broglie wavelength at ϕ_ζ, q is the normalized electron wave amplitude, $\phi = V/\phi_\zeta$ is the normalized electric potential V, $\phi_g = e\phi_\zeta/E_H$, and V_{xc} is the electron exchange-correlation potential (in terms of the Hartree energy $E_H = 27.2$ eV) under the Kohn-Sham local density approximation [17]. The boundary conditions for Eqs. (4) and (5) are $q(1) = \sqrt{2\mu_\zeta/3}$, $q'(1) = 0$, $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = c_\zeta \leq 0$, and $\phi'(1) = a_\zeta \geq 0$. Here, the parameter a_ζ is related to ζ, ϕ_ζ, D, and V_g by the continuous electric field at the beam front, which is

$$a_\zeta \frac{\phi_\zeta}{\zeta} = \frac{V_g - \phi_\zeta}{D - \zeta}. \tag{6}$$

By setting $\phi_\zeta = V_g$ and $\zeta = D$ in Eq. (3), the gap transit time in the quantum regime is $T_{\mathrm{QCL}} = \frac{3}{4}[(a_0 - c_0)/\mu_0]T_{\mathrm{CL}}$, where $T_{\mathrm{CL}} = 3D/\sqrt{2eV_g/m_e}$ is the gap transit time of the 1D classical CL law, and (μ_0, a_0, c_0) are the corresponding values of (μ_ζ, a_ζ, c_ζ) calculated by using Eqs. (4) and (5) at the long-pulse limit of $\tau = T_{\mathrm{QCL}}$. By normalizing the pulse length τ with the transit time T_{QCL}, Eq. (3) gives

$$\frac{\bar{\zeta}}{X_{\mathrm{CL}}\bar{\phi}_\zeta^{1/2}} = \frac{\mu_\zeta}{\mu_0}\frac{a_0 - c_0}{a_\zeta - c_\zeta}, \tag{7}$$

$$\Gamma_Q \equiv \frac{J_{\mathrm{QCL}}}{J_{\mathrm{CL}}} = \mu_\zeta \frac{\bar{\phi}_\zeta^{3/2}}{\bar{\zeta}^2}, \tag{8}$$

where $X_{\mathrm{CL}} = \tau/T_{\mathrm{QCL}} \leq 1$ is the normalized pulse length, $\bar{\phi}_\zeta = \phi_\zeta/V_g$ is the normalized potential at the beam front, $\bar{\zeta} = \zeta/D$ is the normalized position of the beam front, and Γ_Q is the normalized short-pulse quantum CL law in terms of the 1D classical steady-state CL law.

To determine Γ_Q as a function of D, V_g, and τ, we first calculate μ_0, a_0, and c_0 at the long-pulse limit ($X_{\mathrm{CL}} = 1$) from Eqs. (4) and (5). For a finite value of $X_{\mathrm{CL}} < 1$, Eqs. (4)–(7) are solved numerically for μ_ζ, a_ζ, c_ζ, ζ, and ϕ_ζ, to obtain a maximum value of Γ_Q defined in Eq. (8), for which no solutions can be found at other values larger than this maximum value. When quantum effects are negligible [i.e., $\lambda \gg 1$ in Eq. (4)], the solutions are $a_\zeta = a_0 = 4/3$, $c_\zeta = c_0 = 0$, $\mu_\zeta = \mu_0 = 1$, $T_{\mathrm{QCL}} = T_{\mathrm{CL}}$, $\bar{\phi}_\zeta^{1/2} = 2(1 - \sqrt{1 - 3X_{\mathrm{CL}}^2/4})/X_{\mathrm{CL}}$, $\bar{\zeta} = 2(1 - \sqrt{1 - 3X_{\mathrm{CL}}^2/4})$, and Eq. (8) recovers the classical short-pulse CL law, as shown in Eq. (2).

To illustrate the importance of quantum effects at the short-pulse CL law, we first consider the quantum effects of an ultrashort-pulse electron beam ($X_{\mathrm{CL}} \ll 1$) in a classical (large) gap of $V_g = 1$ to 10 kV, and $D = 0.1$ to 1 cm.

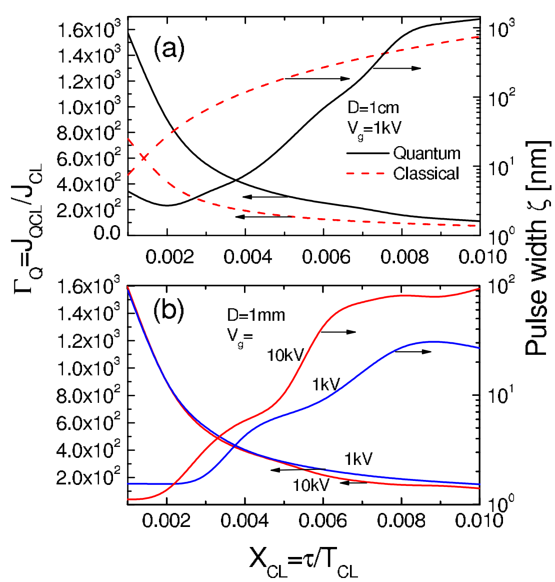

FIG. 1 (color online). The normalized short-pulse quantum CL law in terms of the classical CL law (Γ_Q), and the pulse width of the electron sheet (ζ [nm]), as a function of $X_{\mathrm{CL}} = \tau/T_{\mathrm{CL}}$ in a large gap for (a) $D = 1$ cm and $V_g = 1$ kV, and (b) $D = 1$ mm at $V_g = 1$ and 10 kV. The dashed lines in (a) are the short-pulse classical CL law.

In this case, the quantum effects are not significant at the long-pulse limit ($X_{CL} \approx 1$). In Fig. 1, the normalized short-pulse quantum CL law (Γ_R), and the pulse width (ζ [nm]) are plotted as a function of $X_{CL} = 0.001$ to 0.01, which corresponds to a pulse length of about $\tau = 50$ fs to 16 ps, depending on the values of V_g and D. Here, the values are chosen to have a dc applied electric field in the range of 0.1 to 10 MV/m, which is typical for photoinjectors and field emission based cathodes. It is clear that the classical short-pulse CL law (dashed lines in Fig. 1(a)) is no longer valid when the pulse width ζ is in nanometer scale, and the pulse length is less than 1 ps.

Similar quantum effects may also occur in a nanogap with low voltage, where the quantum effects are important for finite values of $X_{CL} \leq 1$, even at the long-pulse limit. In Fig. 2, Γ_Q is plotted as a function of $X_{CL} = 0.01$ to 1 for $V_g = 1$ to 100 V, and $D = 1$ to 100 nm. In comparison to the classical results (dashed lines), the quantum effects are more significant at small values of X_{CL}, V_g, and D. At $X_{CL} < 0.1$, both normalized quantum and classical short-pulse CL law increase to the inverse power of the normalized pulse length ($\Gamma_Q \propto X_{CL}^{-1}$), but the quantum model is enhanced by a large factor due to the effects of electron tunneling through the space charge field.

It is obvious that if the gap voltage is higher than the electron rest mass, like $U = eV_g/mc^2 \geq 1$, relativistic effects of the electron flows can not be ignored. While 1D steady-state relativistic CL law was first derived decades ago [18], the short-pulse effects of the relativistic CL law has never been studied. Here, we develop a simple 1D short-pulse relativistic CL law without including the effects of self-magnetic field (as it will require a 2D model). The model is found to agree very well with particle-in-cell (PIC) simulation up to $U = 10$ (see below).

Using the same methodology used in the quantum SCL model, we solve the equivalent relativistic SCL model (but ignoring the quantum effects) to obtain the short-pulse relativistic CL law, which is

$$J_{RCL} \equiv \frac{\epsilon_0 mc^3}{2e}\frac{G^2(\gamma_\zeta)}{\zeta^2} = \frac{\epsilon_0}{\tau} a_\zeta \frac{\phi_\zeta}{\zeta}, \tag{9}$$

$$a_\zeta = \frac{G(\gamma_\zeta)(\gamma_\zeta - 1)^{1/4}}{\gamma_\zeta - 1}, \tag{10}$$

where, a_ζ and $\gamma_\zeta = 1 + e\phi_\zeta/mc^2$ are, respectively, the normalized electric field and the relativistic factor at the beam front $x = \zeta$ with a potential of ϕ_ζ, and $G(u) = \int_1^u (r^2 - 1)^{-1/4} dr$ is an integral of dummy variable u. Note in the derivation, we had assumed that the electric field at the cathode is zero at the SCL condition. By setting

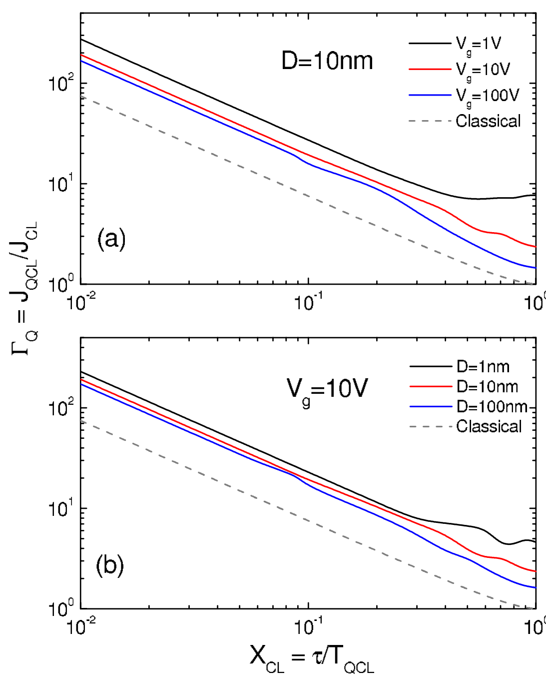

FIG. 2 (color online). The normalized short-pulse quantum CL law in terms of the classical CL law (Γ_Q) as a function of $X_{CL} = \tau/T_{QCL}$ in a nano gap for (a) $D = 10$ nm at $V_g = 1$ to 100 V (top to bottom), and (b) $V_g = 10$ V at $D = 1$ to 100 nm (top to bottom). The dashed lines are the short-pulse classical CL law.

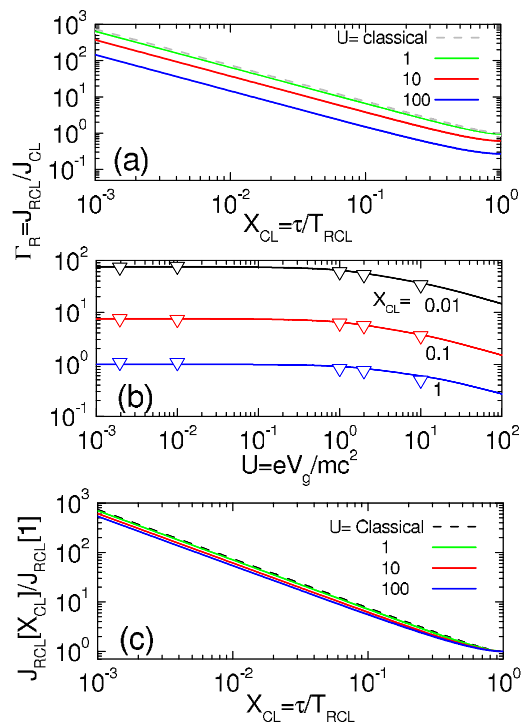

FIG. 3 (color online). The normalized short-pulse relativistic CL law in terms of the classical CL law (Γ_R) as a function of (a) $X_{CL} = \tau/T_{RCL}$ for $U = 1$ to 100 (top to bottom), and (b) U for $X_{CL} = 0.01$ to 1 (top to bottom). The symbols in (b) are the PIC simulation results. (c) The enhancement of the short-pulse relativistic SCL current density over the long-pulse limit $J_{RCL}[X_{CL}]/J_{RCL}$ [1] as a function of X_{CL} for $U = 1$ to 100 (top to bottom). The dashed lines are the short-pulse classical CL law.

$\phi_\zeta = V_g$ and $\zeta = D$ in Eq. (9), the transit time across the gap for the relativistic SCL electron flows is $T_{\rm RCL} = 2(D/c)(\gamma_0^2 - 1)^{1/4}/G(\gamma_0)$, where $\gamma_0 = 1 + eU$, and $U = eV_g/mc^2$. With the normalized pulse length $X_{\rm CL} = \tau/T_{\rm RCL} \leq 1$, Eq. (9) gives

$$\frac{\bar{\zeta}\bar{\phi}_\zeta^{1/4}}{X_{CL}} = \frac{G(\gamma_\zeta)}{G(\gamma_0)}\left(\frac{1+\gamma_0}{1+\gamma_\zeta}\right)^{1/4}, \qquad (11)$$

which is equivalent to Eq. (7) derived in the quantum model. The normalized short-pulse relativistic CL law (in terms of the classical CL law) is

$$\Gamma_R \equiv \frac{J_{\rm RCL}}{J_{\rm CL}} = \frac{9}{8\sqrt{2}}\frac{G^2(\gamma_\zeta)}{(\gamma_0 - 1)^{3/2}\bar{\zeta}^2}, \qquad (12)$$

which recovers the classical limit at $U \ll 1$. Thus, by solving Eqs. (6), (10), and (11), we may calculate Γ_R as a function of $X_{\rm CL}$ and U.

Figure 3 shows the calculated Γ_R as a function of $X_{\rm CL}$ up to $U = 100$, where the classical limits ($U \ll 1$) are plotted in dashed lines. For a given U, Γ_R increases with decreasing value of $X_{\rm CL}$, which also scales as $X_{\rm CL}^{-1}$ at small $X_{\rm CL}$. A 2D PIC simulation code called MAGIC2D [19] has been used to verify the relativistic short-pulse CL law as shown in Fig. 3(b). The simulations were performed by using the over-injection method in a planar diode with a gap separation of 1 cm and an electrode length of 20 cm. A finite pulse τ of current density J is injected into the gap, and the value of J is increased until the formation of a virtual cathode, which causes the reflection of electrons to the cathode. The comparison shows good agreements up to $U = 10$, and the errors are within 5%. At $U > 10$, the self-magnetic field of the relativistic electron flows becomes important that our 1D short-pulse CL model is no longer valid. The calculated results are also plotted in terms of the long-pulse relativistic CL law $J_{\rm RCL}[X_{\rm CL} = 1]$ in Fig. 3(c), which shows clearly that the relativistic effects at $U \geq 1$ will decrease the enhancement due to short-pulse effects.

In conclusion, a 1D short-pulse model of quantum and relativistic Child-Langmuir law have been developed. It is found that the enhancement of the short-pulse CL law over the long-pulse CL law is proportional to the inverse power of the normalized pulse length in all regimes (classical, quantum and relativistic), when the pulse length is much smaller than the gap transit time. Thus, the new scaling of the short-pulse SCL current density is V_g, D^{-1} and τ^{-1}, as compared to the classical steady-state CL law with a scaling of $V_g^{3/2}$ and D^{-2}. Quantum effects are important when the pulse length is ultrashort (like <1 ps) when the size of electron pulse is comparable to the electron de Broglie wavelength, and thus the classical CL current density is increased by a large factor due to electron tunneling through the space charge field. Relativistic effects can not be ignored when the applied voltage is comparable to the electron rest mass, which decreases the enhancement of short-pulse CL law. The transition from the quantum and relativistic models to the classical limits is demonstrated.

It is worth to note that the simple short-pulse quantum CL law presented here is strictly a 1D model, where other practical issues in a realistic photoinjector design are ignored, such as complicated electrode geometry and external focusing magnetic fields. However, the scaling laws developed here may be used as a first estimate in the design or even as an emission algorithm in the traditional gun codes for the development of femtoseond laser-driven high current photocathodes in FEL and other applications, where the quantum effects of such high current ultrashort electron bunches become important. A particle-in-cell code capable of quantum effects is also desired to verify the calculated results.

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[1] P. G. O'Shea and H. P. Feund, Science **292**, 1853 (2001).
[2] T. Plettner, R. L. Byer, E. Colby, and B. Cowan *et al.*, Phys. Rev. Lett. **95**, 134801 (2005).
[3] V. A. Lobastov, R. Srinivasan, and A. H. Zewail, Proc. Natl. Acad. Sci. U.S.A. 102, 7069 (2005).
[4] M. Merano, S. Sonderegger, A. Crottini, and S. Collin *et al.*, Nature (London) **438**, 479 (2005).
[5] B. J. Siwick, J. R. Dwyer, R. E. Jordan, and R. J. D. Miller, Science **302**, 1382 (2003).
[6] P. Hommelhoff, C. Kealhofer, and M. A. Kasevich, Phys. Rev. Lett. **97**, 247402 (2006).
[7] P. Hommelhoff, Y. Sortais, A. A. Talesh, and M. A. Kasevich, Phys. Rev. Lett. **96**, 077401 (2006).
[8] K. L. Jensen, D. W. Feldman, N. A. Moody, and P. G. O'Shea, J. Appl. Phys. **99**, 124905 (2006).
[9] A. Valfells, D. W. Feldman, M. Virgo, P. G. O'Shea, and Y. Y. Lau, Phys. Plasmas **9**, 2377 (2002).
[10] C. D. Child, Phys. Rev. (Series I) **32**, 492 (1911).
[11] J. W. Luginsland, Y. Y. Lau, and R. M. Gilgenbach, Phys. Rev. Lett. **77**, 4668 (1996).
[12] Y. Y. Lau, Phys. Rev. Lett. **87**, 278301 (2001).
[13] R. J. Umstattd and J. W. Luginsland, Phys. Rev. Lett. **87**, 145002 (2001).
[14] A. Rokhlenko and J. L. Lebowitz, Phys. Rev. Lett. **91**, 085002 (2003).
[15] L. K. Ang, T. J. T. Kwan, and Y. Y. Lau, Phys. Rev. Lett. **91**, 208303 (2003).
[16] L. K. Ang, W. S. Koh, Y. Y. Lau, and T. J. T. Kwan, Phys. Plasmas **13**, 056701 (2006).
[17] J. P. Perdew and Y. Wang, Phys. Rev. B **45**, 13244 (1992).
[18] H. R. Jory and A. W. Trivelpiece, J. Appl. Phys. **40**, 3924 (1969).
[19] B. Goplen, L. Ludeking, D. Smithe, and G. Warren, Comput. Phys. Commun. **87**, 54 (1995).