On the evaluation of Pierce parameters C and Q in a traveling wave tube

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A study of an exactly solvable model of a traveling wave tube (TWT) shows that Pierce gain parameter C and space charge parameter Q generally depend on wavenumber k in addition to frequency ω. The choice of k at which C and Q are evaluated may strongly affect their values and, consequently, the values of the small signal gain obtained from 3- and 4-wave Pierce theory. In order to illustrate this effect, we calculate the spatial amplification rate, k_i, from the exact dispersion relation for a dielectric TWT model which is exactly solvable. We compare this exact value of k_i with approximate values obtained from Pierce's classical 3-wave and 4-wave dispersion relations, obtained by making various assumptions on k in the evaluation of C and Q. We find that the various ways to approximate C and Q will have a significant influence on the numerical values of k_i. For our dielectric TWT example, Pierce's 4-wave TWT dispersion relation generally yields the most accurate values of k_i if Q is evaluated for $k = \omega/v_0$, where v_0 is the beam velocity, and if the complete frequency and wavelength dependence of C is retained. Pierce's 3-wave theory also yields accurate values of k_i using a different form of Q from the 4-wave theory. The implications of this result for TWT design are explored. *Published by AIP Publishing.*
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I. INTRODUCTION

For over sixty years, Pierce's classical 3-wave small signal theory of traveling wave tube (TWT) operation laid the foundation for TWT design.[1] This is the case because the small signal, 3-wave theory provides an adequate description of the electron-circuit interaction over approximately 85% of the tube length, even when the TWT is driven to saturation. The three waves include the forward circuit wave, and the fast and slow waves on the beam. The small signal gain, or the spatial amplification rate of the amplifying wave, k_i, is determined from the 3-wave dispersion relation. This dispersion relation is a cubic polynomial, and is fully described by Pierce's four (4) dimensionless parameters, C, b, Q, and d, which, respectively, characterize the TWT gain, detune, "space charge effect," and cold-tube circuit loss.[1–3] These four parameters may also be used to characterize many other coherent radiation sources such as the free electron laser,[4] gyrotron traveling wave amplifier,[5] Smith-Purcell,[6] and metamaterials amplifier.[7] Many earlier works on these sources made an attempt to cast the dispersion relation in Pierce's form; the well-known $I^{1/3}$ scaling in small signal gain for a free electron laser and for a gyrotron are entirely analogous to the scaling in C for TWT, as C is proportional to the cube root of the beam current, I. From Pierce's 3-wave formulation, the concept of launching loss also emerged.[1–3]

The 3-wave theory is an approximation to the 4-wave theory of a TWT, also due to Pierce;[1] the additional wave is a reverse propagating wave on the slow wave circuit. The 4-wave dispersion relation is a fourth degree polynomial in the complex wave number k and is also fully characterized by the four dimensionless parameters, C, b, Q, and d. This reverse propagating mode can contribute to oscillation in a TWT in the presence of reflections,[8–10] but is generally unimportant in characterizing the forward wave gain. For calculation of forward wave gain, the backward propagating circuit mode is usually ignored. Since Pierce's 3-wave theory is an approximation to the 4-wave dispersion relation, we shall mainly concentrate on the formulation of C and Q according to the 4-wave theory. The proper formulation of C and Q, based on Pierce's 3-wave theory, will be given in Section III B.

Among the four parameters (C, b, Q, and d) in Pierce's theory, the most difficult to evaluate reliably is the "space charge parameter," Q.[11,12] While Q is not as crucial as the gain parameter (C) and the detune parameter (b), it does affect the accurate determination of k_i, the spatial exponentiation rate of the amplifying wave amplitude. This is an issue because a few percent change in k_i would result in a very significant change in the predicted power gain, which is roughly given by $\exp(2k_iL)$, where L is the interaction length. How to calculate Q reliably remains an open question for a general slow wave structure.[12] This uncertainty is thus also present in

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various simulation codes that need to include the "space charge effects."[13] For helix, coupled cavity, and folded-waveguide TWTs, one method[2,11,13,14] used to calculate Q is to consider the beam's electrostatic space charge wave, assuming that the slow wave structure is replaced by a uniform, cylindrical waveguide. This procedure yields good agreement with experimental measurements and with particle-in-cell simulations[15] in many, but not all, cases. In this paper, we examine various approximations to Q (Refs. 11, 12, 14, and 16) and C and compare the resulting exponentiation rates with the exact values for a TWT model for which the exact dispersion relation may be obtained. We shall first concentrate on Pierce's 4-wave theory, and its reduction to the 3-wave theory.

The model we shall consider is a dielectric TWT.[12,17] Despite its impracticality, it has yielded some interesting insight into the absolute instability near the band edges of a TWT.[17] We shall use it here for the exact evaluation of Pierce's parameter, Q, so as to understand the nature of various approximations of Q.

II. DIELECTRIC TWT

Consider a monoenergetic, nonrelativistic electron sheet beam of surface charge density σ_0, propagating in the z-direction in the mid-plane of a smooth dielectric, planar transmission line at speed v_0 and guided by an infinite magnetic field. In a small signal theory, assuming $e^{j(\omega t-kz)}$ dependence for all quantities, the beam–wave interaction in this dielectric TWT is governed by the *exact* dispersion relation[12,17]

$$(\omega - kv_0)^2 = \omega_a^2 Z(\omega,k), \tag{1}$$

where

$$Z(\omega,k) = -(pa)\tan(pa). \tag{2}$$

In Eqs. (1) and (2), $Z(\omega, k)$ is the normalized "plasma reduction factor," which is also known as the normalized "wave impedance," $\omega_a^2 = |e\sigma_0|/2m\varepsilon a$, $p^2 = \omega^2\varepsilon\mu_0 - k^2$, where ε is the dielectric constant of the planar transmission line of separation $2a$, and μ_0 is the free space permeability. The cold tube dispersion relation is given by $Z(\omega,k)\to\infty$, or $pa = (n-1/2)\pi$, where $n = 1, 2, 3,\ldots$; the cutoff frequency of the n-th mode is $\omega_{cn} = (n-1/2)\pi c_\varepsilon/a$ with $c_\varepsilon = 1/\sqrt{\varepsilon\mu_0}$. Figure 1 shows two possible operating points, P_1 and P_2, at which the beam mode is synchronous with the circuit mode, i.e., $v_0 = v_{ph} = \omega/k$. In this paper, we assume that there is no cold tube loss, so that Pierce's loss parameter d = 0.

The reduction factor in Eq. (2) may also be written without any approximation[12] as

$$Z(\omega,k) = 2\sum_{n=1}^{\infty}\frac{\omega^2 - k^2c_\varepsilon^2}{\omega^2 - k^2c_\varepsilon^2 - \omega_{cn}^2} = \frac{2\omega_{c1}^2}{\omega^2 - k^2c_\varepsilon^2 - \omega_{c1}^2} + R(\omega,k), \tag{3}$$

where

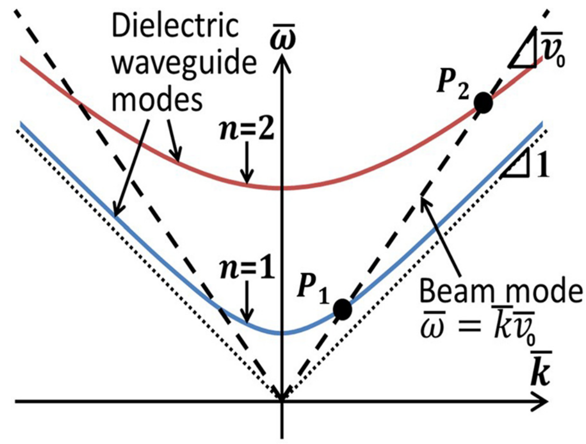

FIG. 1. The normalized dispersion diagram for the first two waveguide modes (n = 1 and n = 2) and the beam mode in a dielectric TWT, with possible operating points at P_1 and P_2. This paper considers operation at P_1 and is normalized accordingly: $\bar{\omega} = \omega/\omega_{c1}$, $\bar{k} = kc_\varepsilon/\omega_{c1}$, and $\bar{v}_0 = v_0/c_\varepsilon$ (>1). The cutoff frequency is at $\bar{\omega} = 1$, and the slope of the dielectric light line is unity.

$$R(\omega,k) = 2 + 2\sum_{n=2}^{\infty}\frac{\omega^2 - k^2c_\varepsilon^2}{\omega^2 - k^2c_\varepsilon^2 - \omega_{cn}^2} = 2 + 2\sum_{n=2}^{\infty}\frac{\bar{\omega}^2 - \bar{k}^2}{\bar{\omega}^2 - \bar{k}^2 - (2n-1)^2} \equiv R(\bar{\omega},\bar{k}) \tag{4}$$

and we have defined the dimensionless frequency and wavenumber as $\bar{\omega} = \omega/\omega_{c1}$, and $\bar{k} = kc_\varepsilon/\omega_{c1}$. In deriving Eq. (3), we have used the identity $-x\tan x = 2\sum_{n=1}^{\infty} x^2/[x^2 - (n-1/2)^2\pi^2]$. See, also, Appendix A of Ref. 12 for details. The middle expression of Eq. (3) shows the individual contribution to Z from each vacuum circuit mode, whereas the last expression separates the term representing the synchronous interaction of the fundamental waveguide mode (n = 1) when the operating point is close to P_1 in Fig. 1, as we will assume here and below. The remainder term, R(ω,k) in Eq. (3), accounts for the non-synchronous interactions involving all higher order modes.[1,12] R, is often called the "AC space charge term."[1–3,12] In fact, R is proportional to Pierce's space charge parameter, Q [see Eq. (11)].

We use Eq. (3) to re-write the exact dielectric-TWT dispersion relation, Eq. (1), in dimensional form in Eq. (5a), and in dimensionless form in Eq. (5b)

$$[(\omega - kv_0)^2 - \omega_{c1}^2 sR(\omega,k)](\omega^2 - k^2c_\varepsilon^2 - \omega_{c1}^2) = 2s\omega_{c1}^4, \quad \textbf{Exact}, \tag{5a}$$

$$\left[(\bar{\omega} - \bar{k}\bar{v}_0)^2 - sR(\bar{\omega},\bar{k})\right](\bar{\omega}^2 - \bar{k}^2 - 1) = 2s. \quad \textbf{Exact}. \tag{5b}$$

Here, $s = \omega_a^2/\omega_{c1}^2$ is a dimensionless parameter proportional to the beam current, $\bar{\omega} = \omega/\omega_{c1}$, $\bar{k} = kc_\varepsilon/\omega_{c1}$, and $\bar{v}_0 = v_0/c_\varepsilon$ (>1). The usual interpretation of Eqs. (5a) or (5b) is that the beam mode (represented by the square brackets) is coupled to the fundamental circuit mode (represented by the parentheses), and the coupling constant is proportional to s (or C^3). Once we fix the dimensionless constants $\bar{v}_0$ and s, we may calculate from Eq. (5b) the normalized spatial amplification rate, $\mathrm{Im}(\bar{k}) = \bar{k}_i(\bar{\omega})$, as a function of the normalized (real-valued) frequency $\bar{\omega}$. In the numerical examples below, we set $\bar{v}_0 = 1.2$.

In the absence of cold tube loss, d = 0, Pierce's 4-wave dispersion relation is[1]

$$[(\omega - kv_0)^2 - 4QC^3(kv_0)^2](\omega^2 - k^2v_{ph}^2) = 2C^3\omega^2k^2v_0v_{ph}, \quad \textbf{Pierce's 4-wave}, \quad (6)$$

where C is the dimensionless gain parameter, Q is the space charge parameter, and v_{ph} is the phase velocity of the cold circuit mode at frequency ω. The first term in Eq. (6) represents the two beam modes, and the second term represents both forward and reverse-propagating circuit modes, $\omega = \pm kv_{ph}$. In Eq. (6), all parameters C and Q are usually assumed to depend on the frequency ω, but *not* on the wavenumber k. Under this assumption, Eq. (6) is a 4th order polynomial in k. The four roots $k(\omega)$ of Eq. (6) and, in particular, the spatial amplification rate, $k_i(\omega)$, depend on the values of C and Q, which can be difficult or impossible to compute from first principles for a general slow wave structure.

To calculate the gain in the forward wave, Pierce ignored the reverse propagating circuit mode and approximated Eq. (6), as the 3-wave TWT dispersion relation that now bears his name[18]

$$[(\omega - kv_0)^2 - 4QC^3\omega^2](\omega - kv_{ph}) = C^3\omega^3(v_{ph}/v_0). \quad \textbf{Pierce's 3-wave}. \quad (7)$$

Equation (7) is commonly written in the form[1–3]

$$(\delta^2 + 4QC)(\delta + jb) = -j, \quad (8)$$

where $\delta = -j(k - k_e)/Ck_e$, $b = (v_0/v_{ph} - 1)/C$, and $k_e = \omega/v_0$. Similar to Eq. (6), Eq. (7) is usually regarded as a cubic polynomial in k, the roots of which yield the three propagation constants, k, of the 3 waves at frequency ω. Pierce did not address the possible distinction in the values of (C, Q) in his 4-wave theory, Eq. (6), and in his 3-wave theory, Eq. (7), nor the proper way of evaluating them.

For the dielectric TWT operating at the fundamental mode (P_1 in Fig. 1), we now write the exact dispersion relation, Eq. (5a), in the form of Pierce's 4-wave dispersion relation, Eq. (6). First, note that $(\omega^2 - k^2c_\varepsilon^2 - \omega_{c1}^2) = (\omega^2 - k^2v_{ph}^2)c_\varepsilon^2/v_{ph}^2$, where v_{ph} is given by

$$\frac{v_{ph}}{c_\varepsilon} = \frac{\omega}{\sqrt{\omega^2 - \omega_{c1}^2}} = \frac{\bar{\omega}}{\sqrt{\bar{\omega}^2 - 1}}. \quad (9)$$

Comparing Eq. (5a) with Pierce's 4-wave dispersion relation, Eq. (6), the exact values of the gain parameter, $C = C(\omega, k)$, and the "space charge parameter," $Q = Q(\omega, k)$, in Eq. (6) may then be identified for the dielectric TWT

$$C^3 = \frac{\pi^2}{4}\left(\frac{\omega_a^2}{\omega^2}\right)\left(\frac{1}{k^2a^2}\right)\left(\frac{v_{ph}}{v_0}\right) = \frac{s}{\bar{v}_0} \times \frac{1}{\bar{k}^2\bar{\omega}\sqrt{\bar{\omega}^2 - 1}}, \quad \textbf{Exact, 4-wave theory}, \quad (10)$$

$$Q = \frac{R(\omega, k)}{4}\left(\frac{\omega^2}{\omega_{c1}^2}\right)\left(\frac{c_\varepsilon^2}{v_0v_{ph}}\right) = \frac{\bar{R}(\bar{\omega}, \bar{k})}{4\bar{v}_0}\bar{\omega}\sqrt{\bar{\omega}^2 - 1}. \quad \textbf{Exact, 4-wave theory}. \quad (11)$$

In the last expressions in Eqs. (10) and (11), both C and Q are expressed in dimensionless variables. It is important to note that both depend on $\bar{\omega}$ and $\bar{k}$. These expressions of C and Q, identified for Pierce's 4-wave theory, will also be used in Pierce's 3-wave dispersion relation, Eq. (7), and the results will be shown in Figs. 3–5.[18]

Given the normalized beam velocity $\bar{v}_0$, the normalized frequency $\bar{\omega}_0$ and normalized wavenumber $\bar{k}_0$ at the intersection point P_1 (Fig. 1) may easily be obtained from the relation $\bar{\omega}_0 = \bar{k}_0\bar{v}_0 = \sqrt{\bar{k}_0^2 + 1}$. For $\bar{v}_0 = 1.2$, we obtain $(\bar{\omega}_0, \bar{k}_0) = (1.80907, 1.50756)$ at which the value of C, denoted as C_0, is related to the normalized beam current s by

$$C_0^3 = 0.134444s, \qquad s = 7.43805\, C_0^3 \quad (12)$$

according to Eq. (10). Henceforth, we will use C_0, instead of s, to designate the beam current. We note that Pierce's gain parameter, C, typically lies between 0.01 and 0.15.

III. COMPARISON OF SPATIAL AMPLIFICATION RATES

A. C and Q based on the 4-wave theory

The expressions for C and Q, given by Eqs. (10) and (11), were obtained for the 4-wave theory, Eq. (6). Both C and Q depend on ω and k. To solve for the complex wavenumber k from the fourth (third) degree polynomial according to Pierce's 4-wave (3-wave) theory, an approximation must be made for the value of k at which C and Q are evaluated, in Eqs. (10) and (11). We first study two such approximations, which we will call I and II, defined by

$$C(\omega, k) = C(\omega, k_{circ}(\omega)) \equiv C_I(\omega), \quad Q(\omega, k) = Q(\omega, k_{circ}(\omega)) \equiv Q_I(\omega), \quad (13a)$$

$$C(\omega, k) = C(\omega, k_{beam}(\omega)) \equiv C_{II}(\omega), \quad Q(\omega, k) = Q(\omega, k_{beam}(\omega)) \equiv Q_{II}(\omega), \quad (13b)$$

where in I [Eq. (13a)], we have approximated k by the (real valued) vacuum circuit mode wavenumber, $k_{circ}(\omega)$. In approximation II [Eq. (13b)], we have approximated k by the (real valued) beam mode wavenumber, $k_{beam} = \omega/v_0$. A third approximation, designated as III [Eq. (13c)], exploits the fact that k^2C^3 is a function of ω alone, according to Eq. (10),

$$C(\omega, k) = C_{III}(\omega, k), \quad Q(\omega, k) = Q(\omega, k_{beam}(\omega)) \equiv Q_{III}(\omega), \quad [k^2C^3 \text{ depends only on } \omega]. \quad (13c)$$

In Eq. (13c), Q_{III} is the same as Q_{II} in Eq. (13b).[19] Roots of Eqs. (6) and (7) using the three approximations I, II, and III will now be compared with the roots of the exact dispersion relation, Eq. (1).

For $\bar{v}_0 = 1.2$ and $C_0 = 0.08$, so that $s = 0.003808$ [cf. Eq. (12)], we plot in Fig. 2(a) the normalized amplification rate, $\bar{k}_i = \mathrm{Im}(\bar{k})$, according to the exact dispersion relation, Eq. (5b). Also shown are the results from Pierce's 4-wave theory, Eq. (6), in which we use approximations Eqs. (13a)–(13c) for Q and C. Note that the exponentiation rate is most accurately given by Pierce's 4-wave theory using approximation III, even for the relatively high value of $C_0 = 0.08$. The normalized phase velocity, $\bar{\omega}/\bar{k}_r = \omega/k_r c_\varepsilon$, for the amplifying wave according to the exact theory, is shown in Fig. 2(b), along with the normalized beam velocity $\bar{v}_0 = v_0/c_\varepsilon$. We see that the amplifying wave is a slow wave.

To evaluate Pierce's 4-wave theory and 3-wave theory and the use of approximations I, II, and III, we plot in Figs. 3, 4, and 5 the results for approximation I, II, and III, respectively. In each figure, we set $C_0 = 0.01$, 0.02, 0.04, and 0.08. For each C_0, in order to see the effect of Q (in addition to the effects of various approximations to Q), we also include the data with Q set to zero in Eqs. (6) and (7).

In Fig. 3, we assumed k to be the vacuum circuit mode wavenumber, $k_{circ}(\omega)$, in Q and C, and in all terms multiplied by C^3 in Eq. (6). We calculated k_i from Eqs. (6) and (7), with and without the Q term. This shows the importance of the space charge effects, according to both the 3-wave and 4-

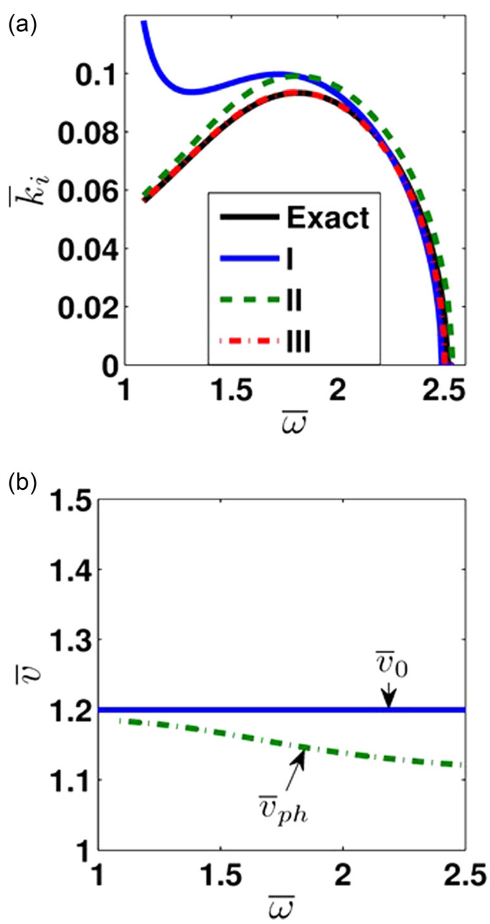

FIG. 2. (a) Top: The value of $\bar{k}_i$ as a function of $\bar{\omega}$ for $C_0 = 0.08$ according to the exact theory and to Pierce's 4-wave theory [Eq. (6)] using approximations I, II, and III. (b) Bottom: The plot of normalized phase velocity, $\bar{v}_{ph} = \bar{\omega}/\bar{k}_r = \omega/k_r c_\varepsilon$, for the amplifying wave, and the normalized beam velocity $\bar{v}_0 = v_0/c_\varepsilon$.

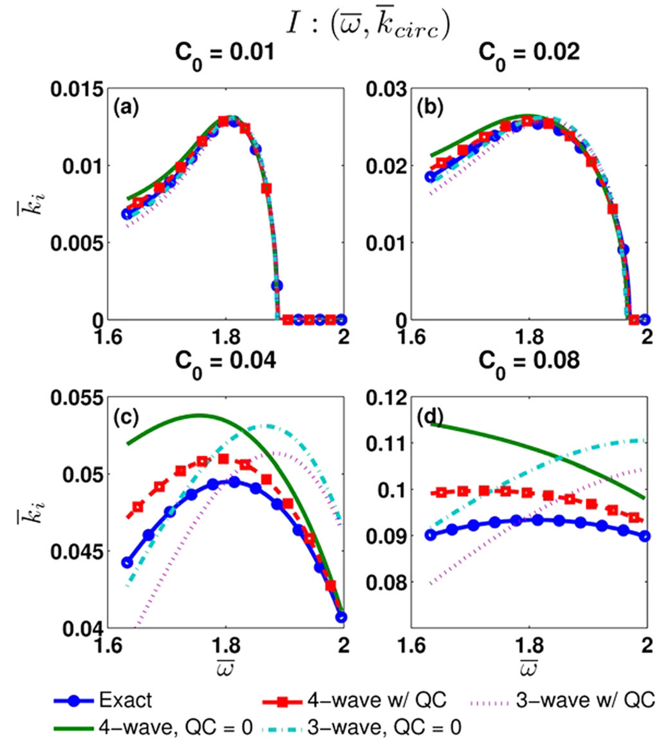

FIG. 3. The value of $\bar{k}_i$ as a function of $\bar{\omega}$ from the exact theory and from Pierce's 3-wave and 4-wave theory using approximation I (Eq. (13a)), for (a) $C_0 = 0.01$, (b) $C_0 = 0.02$, (c) $C_0 = 0.04$, and (d) $C_0 = 0.08$. In each graph, Pierce's theory with and without the effect of Q is shown.

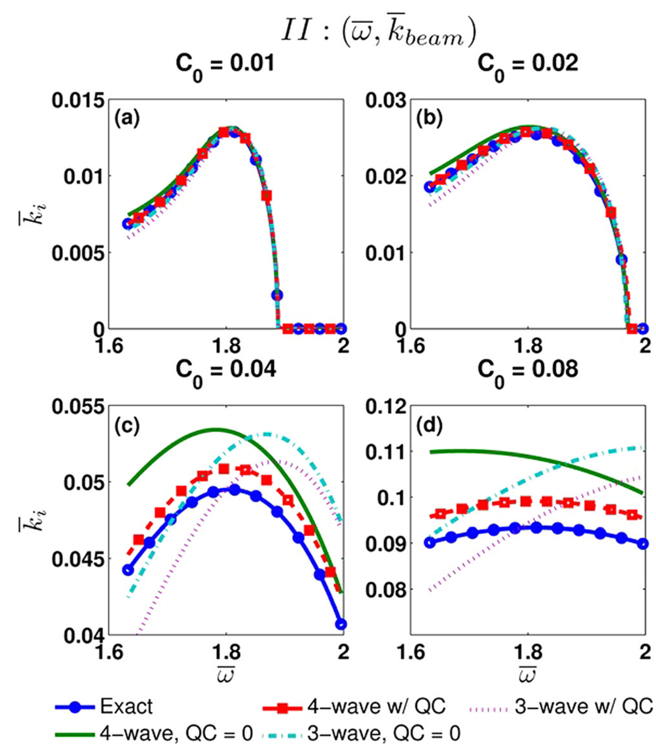

FIG. 4. The value of $\bar{k}_i$ as a function of $\bar{\omega}$ from the exact theory and from Pierce's 3-wave and 4-wave theory using approximation II (Eq. (13b)), for (a) $C_0 = 0.01$, (b) $C_0 = 0.02$, (c) $C_0 = 0.04$, and (d) $C_0 = 0.08$. In each graph, Pierce's theory with and without the effect of Q is shown.

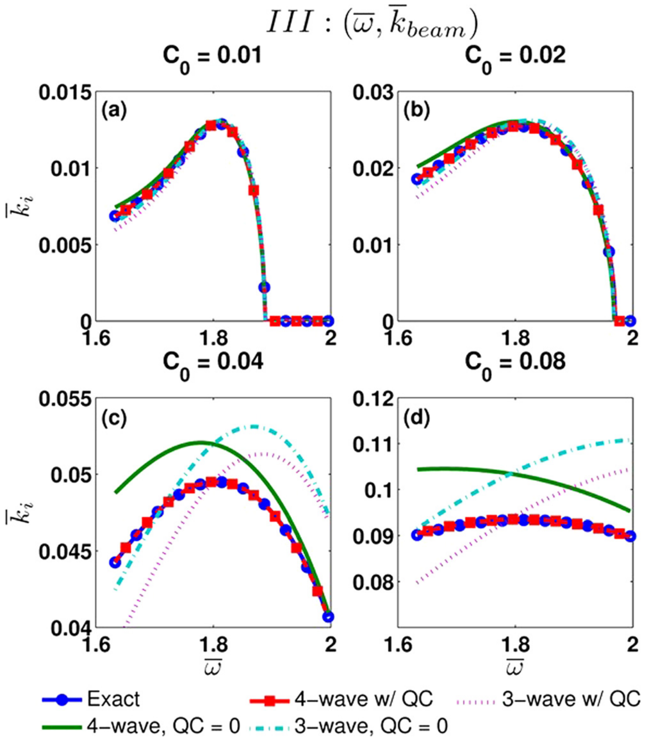

FIG. 5. The value of $\bar{k}_i$ as a function of $\bar{\omega}$ from the exact theory and from Pierce's 3-wave and 4-wave theory using approximation III (Eq. (13c)), for (a) $C_0 = 0.01$, (b) $C_0 = 0.02$, (c) $C_0 = 0.04$, and (d) $C_0 = 0.08$. In each graph, Pierce's theory with and without the effect of Q is shown.

wave theory of Pierce. The value of k_i from the exact dispersion relation is also included for comparison. From Fig. 3(a), we see that for $C_0 = 0.01$, all results show reasonable agreement with the exact theory. This agreement is no longer observed when $C_0 = 0.02$ for low frequency (Fig. 3(b)), and is much worse for $C_0 = 0.04$ and $C_0 = 0.08$, as shown in Figs. 3(c) and 3(d). Note from Figs. 3(c) and 3(d) that the shape of the curves in the 4-wave theory is much closer to the exact theory than the 3-wave theory. The 4-wave theory including the space charge effects seems to give the best fit for approximation I. The 3-wave theory gives poor agreement at high values of C_0, in general.

In Fig. 4, we adopt approximation II, where we assumed k to be the beam mode wavenumber, $k(\omega) = \omega/v_0$, in Q and C, and in all terms multiplied by C^3 in Eq. (6).[19] The results are similar to approximation I, shown in Fig. 3. The 4-wave theory including the space charge effects seems to give the best fit to the exact theory. The 3-wave theory gives poor agreement at high value of C_0, in general.

In Fig. 5, we use approximation III, which is the 4-wave theory [Eq. (6)] in which k^2C^3 depends only on ω, and k is approximated by the beam mode wavenumber, $k(\omega) = \omega/v_0$, in Q.[19] It is shown in Fig. 5 that there is excellent agreement between approximation III and the exact theory, even if C_0 is as high as 0.08. See also Fig. 2(a). The 3-wave theory in Fig. 5 was obtained by assuming k^2C^3 to depend only on ω, and then assuming k to be the beam mode wavenumber, $k(\omega) = \omega/v_0$, in Q and C. Thus, the 3-wave theory in approximation III (Fig. 5) is the same as the 3-wave theory in approximation II (Fig. 4).

B. C and Q based on the 3-wave theory

We may cast the exact dispersion relation, Eq. (1), in the form of Pierce's 3-wave dispersion relation, Eq. (7).[18] In doing so, the expressions for C and Q in this 3-wave representation, designated as C_3 and Q_3, respectively, would be different from Eqs. (10) and (11). We found that C_3 and Q_3 are given by

$$C_3^3 = \frac{s\bar{v}_0}{\bar{\omega}^3\sqrt{\bar{\omega}^2 - 1}}, \quad \textbf{Exact, 3-wave theory}, \tag{14}$$

$$4Q_3C_3^3 = \frac{s}{\bar{\omega}^2}R_3(\bar{\omega}, \bar{k}), \quad \textbf{Exact, 3-wave theory}, \tag{15}$$

$$R_3(\bar{\omega}, \bar{k}) = \frac{1}{(\bar{\omega}^2 - 1)\left(1 + \bar{k}/\sqrt{\bar{\omega}^2 - 1}\right)} + R(\bar{\omega}, \bar{k}), \quad \textbf{Exact, 3-wave theory}, \tag{16}$$

where $R(\bar{\omega}, \bar{k})$ is given by Eq. (4). That is, using (C_3, Q_3) for (C, Q) in the 3-wave equation, (7), Eq. (7) is then identical to the exact dispersion relation, Eq. (1).

Figure 6 shows the values of k_i using (C_3, Q_3) in the 3- and 4-wave dispersion relation, Eqs. (6) and (7). Since C_3 is a function of ω alone by Eq. (14), we apply approximation IV only to Q_3, wherein we approximate k by ω/v_0 in Q_3 in Eqs. (6) and (7). The results using approximation IV are shown in Fig. 6, where the exact solution of k_i obtained from Eq. (1) is also displayed. Here, we see that the 3-wave approximation, using C_3 and Q_3, yields excellent agreement for k_i with the exact theory, even for $C_0 = 0.08$ and over a wide range of frequencies, just like the 4-wave theory shown in Fig. 5(d).

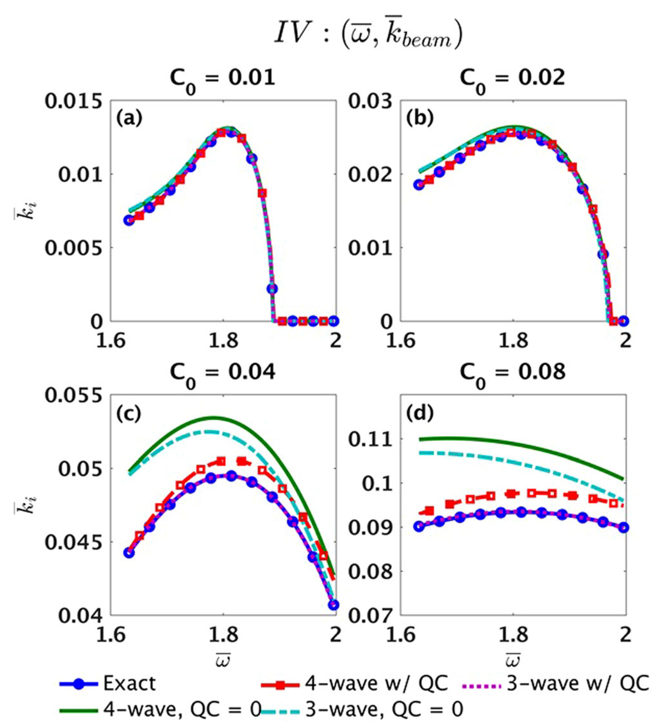

FIG. 6. The value of $\bar{k}_i$ as a function of $\bar{\omega}$ from the exact theory and from Pierce's 3-wave and 4-wave theory using approximation IV applied to Q_3, for (a) $C_0 = 0.01$, (b) $C_0 = 0.02$, (c) $C_0 = 0.04$, and (d) $C_0 = 0.08$. In each graph, Pierce's theory with and without the effect of Q_3 is shown.

IV. CONCLUSION

In general, both Pierce parameters, C and Q, depend on ω and k. Their expressions identified for the 3-wave theory differ from the 4-wave theory. The exactly solvable model of the dielectric TWT illustrates the subtlety of this aspect. It shows that accurate calculations of the exponentiation rate, k_i, may be obtained by approximating k by $k_{beam} = \omega/v_0$ in Q, and by taking the k-dependence on C as completely as possible. A few percent change in k_i will result in a large variation in the output signal, even in the linear theory. If the 4-wave theory needs to be used to account for the effects of the backward wave on the circuit, the formulation of the boundary conditions at both the input and output requires care if this reverse-propagating circuit wave is to be included.[10] The "launching loss" in a 4-wave theory is not immediately known.

While QC is only a "secondary" parameter, compared with C and b, it is central to the threshold condition for backward wave oscillation in TWT.[8] See, also, p. 407 of Gewartowski and Watson.[2]

We are currently analyzing C and QC for a realistic tape helix TWT.[20] The hot-tube solution is constructed with the inclusion of a pencil beam, using the approach of Ref. 20. The numerical results will be obtained and compared with Pierce's 3-wave and 4-wave theory. For a sine-waveguide TWT, QC was inferred from a particle-in-cell simulation,[21] as the analytic formulation is unavailable for the complicated geometry.

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[19] In approximations II and III, we assumed that k is approximated by the beam mode wavenumber, $k_{beam}(\omega) = \omega/v_0$ in Q [cf. Eqs. (6) and (11)]. Another approximation is to expand Q as a function of k, to first order in $(k - \omega/v_0)$, and this is essentially Dialetis' method.[16] We have shown that there is little difference in the calculated values of k_i between Dialetis' method and by assuming $k_{beam}(\omega) = \omega/v_0$ in Q for the dielectric TWT, for the parameter ranges studied.
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